Asymptotic silence of generic cosmological singularities

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In this letter we investigate the nature of generic cosmological singularities using the framework developed by Uggla *et al.* We do so by studying the past asymptotic dynamics of general vacuum G_2 cosmologies, models that are expected to capture the singular behavior of generic cosmologies with no symmetries at all. In particular, our results indicate that asymptotic silence holds, i.e., that particle horizons along all timelines shrink to zero for generic solutions. Moreover, we provide evidence that spatial derivatives become dynamically insignificant along generic timelines, and that the evolution into the past along such timelines is governed by an asymptotic dynamical system which is associated with an invariant set — the silent boundary. We also identify an attracting subset on the silent boundary that organizes the oscillatory dynamics of generic timelines in the singular regime. In addition, we discuss the dynamics associated with recurring spike formation.

The singularity theorems of Penrose and Hawking [1] state that generic cosmological models contain an initial singularity, but do not give any information on the nature of this singularity. Heuristic investigations of this issue led Belinskiĭ, Khalatnikov and Lifshitz [2] (BKL) to propose that a generic cosmological initial singularity is spacelike, local and oscillatory. Uggla *et al.* [3] (UEWE) reformulated Einstein's field equations (EFEs) by introducing scale-invariant variables which have the property that all individual terms in EFEs become asymptotically bounded, for generic solutions. This made it possible to characterize a generic cosmological initial singularity in terms of specific limits. The numerical study of the picture proposed by UEWE was initiated in Ref. [4], specializing to Gowdy vacuum spacetimes which have a non-oscillatory singularity. This letter presents the results of the first detailed study of the oscillatory asymptotic dynamics of inhomogeneous cosmologies from the dynamical systems point of view introduced in UEWE.

Here we focus on vacuum cosmologies with an Abelian symmetry group G_2 with two commuting spacelike Killing vector fields, and the spatial topology of a 3-torus. This is arguably the simplest class of inhomogeneous models that is expected to capture the properties of a generic oscillatory singularity. Numerical investigations of G_2 spacetimes supporting the BKL proposal were carried out by Weaver *et al.* in Ref. [5, 6].

UEWE used an orthonormal frame formalism and factored out the expansion of a timelike reference congruence $\boldsymbol{e}_0$ by normalizing the dynamical variables with the isotropic Hubble expansion rate H of $\boldsymbol{e}_0$. This yielded a dimensionless state vector $\boldsymbol{X} = (E_\alpha{}^i) \oplus \boldsymbol{S}$, where $E_\alpha{}^i$ are the Hubble-normalized components of the spatial frame vectors orthogonal to $\boldsymbol{e}_0$; $\boldsymbol{e}_\alpha = e_\alpha{}^i \partial_i$, $E_\alpha{}^i = e_\alpha{}^i/H$.

The approach to an initial singularity will be said to be *asymptotically silent* for timelines along which $E_\alpha{}^i \to 0$, and *asymptotically silent and local* for timelines along which $E_\alpha{}^i \to 0$ and $E_\alpha{}^i \partial_i \boldsymbol{S} \to 0$; in the latter case $E_\alpha{}^i = 0$ defines an unphysical invariant set, the silent boundary. [In UEWE the concept of a "silent singularity" was defined. However, the possibility of "recurring spike formation," discussed below, motivates the present distinctions and definitions.] The evolution equations for $\boldsymbol{S}$ on the silent boundary are identical to EFEs for spatially self-similar (SSS) and spatially homogeneous (SH) models, in a symmetry adapted Hubble-normalized orthonormal frame [4].

Motivated by the discussion in UEWE, we conjecture that $\mathcal{U}_{\mathrm{vac}}^-$, the union of the bounded vacuum SH Type–I (Kasner) and SH Type–II subsets on the silent boundary, form an attracting subset that organizes the oscillatory dynamics of generic timelines approaching an asymptotically silent and local vacuum singularity.

To obtain the equations for vacuum G_2 cosmologies, we introduce coordinates $\{t, x, y_1, y_2\}$ and an orthonormal frame: $\boldsymbol{e}_0 = N^{-1} \partial_t$, $\boldsymbol{e}_1 = e_1{}^1 \partial_x + e_1{}^2 \partial_{y_1} + e_1{}^3 \partial_{y_2}$, $\boldsymbol{e}_2 = e_2{}^2 \partial_{y_1}$ and $\boldsymbol{e}_3 = e_3{}^2 \partial_{y_1} + e_3{}^3 \partial_{y_2}$, cf. Ref. [7]; N and $e_\alpha{}^i$ are functions of t and x only. For comparison with previous work, $\boldsymbol{e}_2$ is aligned with the Killing vector field ∂_{y_1}. We choose 2π-periodic coordinates x, y_1 and y_2, yielding a spatial 3-torus topology, and a temporal gauge such that the area density of the G_2-orbits is given by $\mathcal{A} := (e_2{}^2 e_3{}^3)^{-1} \propto e^{-t}$; this is convenient since the level sets of $\mathcal{A}$ give a global foliation for maximally globally hyperbolic vacuum G_2 cosmologies [8], and since $t \to +\infty$ at the singularity [9]. The G_2 symmetry implies $0 = \boldsymbol{e}_2(f) = \boldsymbol{e}_3(f)$ for any coordinate scalar f. Thus, only $N^{-1} \partial_t$ and $e_1{}^1 \partial_x$ act nontrivially on coordinate scalars, and hence the equations of all spatial frame variables except $e_1{}^1$ decouple; the essential

Hubble-normalized variables are thus $E_1{}^1 := e_1{}^1/H$ and a subset of connection components, which depend on t and x only. Inserting the above restrictions into the relations in App. 5 of UEWE yields $0 = A^\alpha = N_{1\alpha} = N_{33} = \Sigma_{12} = \dot{U}_2 = \dot{U}_3$ ($\alpha = 1, 2, 3$), and the spatial frame gauge $R_1 = -\Sigma_{23}$, $R_2 = -\Sigma_{31}$, $R_3 = 0$. In addition, it is convenient to define: $\Sigma_+ := \frac{1}{2}(\Sigma_{22} + \Sigma_{33}) = -\frac{1}{2}\Sigma_{11}$, $\Sigma_- := \frac{1}{2\sqrt{3}}(\Sigma_{22} - \Sigma_{33})$, $\Sigma_\times := \frac{1}{\sqrt{3}}\Sigma_{23}$, $\Sigma_2 := \frac{1}{\sqrt{3}}\Sigma_{31}$, $N_- := \frac{1}{2\sqrt{3}}N_{22}$ and $N_\times := \frac{1}{\sqrt{3}}N_{23}$. The Hubble-normalized variables have the following physical interpretation: Σ_+, Σ_-, $\Sigma_\times$, Σ_2 are shear variables for $\boldsymbol{e}_0$; $\dot{U} = \dot{U}_1$ describes the acceleration of $\boldsymbol{e}_0$; N_- and $N_\times$ are spatial connection components that determine the spatial curvature; R_α yields the angular velocity of the spatial frame $\{\boldsymbol{e}_\alpha\}$. The lapse function is given by $N = -\frac{1}{2}H^{-1}(1 + \Sigma_+)^{-1}$. The deceleration parameter q and the spatial Hubble gradient r are defined by $(q + 1) := -H^{-1}\boldsymbol{\partial}_0 H$ and $r := -H^{-1}\boldsymbol{\partial}_1 H$, respectively, with $\boldsymbol{\partial}_0 := -2(1 + \Sigma_+)\partial_t$ and $\boldsymbol{\partial}_1 := E_1{}^1 \partial_x$. These definitions yield the *integrability condition* $\boldsymbol{\partial}_0 r - \boldsymbol{\partial}_1 q = (q + 2\Sigma_+) r - (r - \dot{U})(q + 1)$.

Imposing the above restrictions and gauge choices on EFEs in vacuum yields the following *evolution equations* and *constraints*:

$$\boldsymbol{\partial}_0 E_1{}^1 = (q + 2\Sigma_+) E_1{}^1 \tag{1a}$$

$$\boldsymbol{\partial}_0(1 + \Sigma_+) = (q - 2)(1 + \Sigma_+) + 3\Sigma_2^2 \tag{1b}$$

$$\boldsymbol{\partial}_0 \Sigma_2 = (q - 2 - 3\Sigma_+ + \sqrt{3}\Sigma_-)\Sigma_2 \tag{1c}$$

$$\boldsymbol{\partial}_0 \Sigma_- + \boldsymbol{\partial}_1 N_\times = (q - 2)\Sigma_- + (r - \dot{U})N_\times$$
$$+ 2\sqrt{3}\Sigma_\times^2 - 2\sqrt{3}N_-^2 - \sqrt{3}\Sigma_2^2 \tag{1d}$$

$$\boldsymbol{\partial}_0 N_\times + \boldsymbol{\partial}_1 \Sigma_- = (q + 2\Sigma_+)N_\times + (r - \dot{U})\Sigma_- \tag{1e}$$

$$\boldsymbol{\partial}_0 \Sigma_\times - \boldsymbol{\partial}_1 N_- = (q - 2 - 2\sqrt{3}\Sigma_-)\Sigma_\times$$
$$- (r - \dot{U} + 2\sqrt{3}N_\times)N_- \tag{1f}$$

$$\boldsymbol{\partial}_0 N_- - \boldsymbol{\partial}_1 \Sigma_\times = (q + 2\Sigma_+ + 2\sqrt{3}\Sigma_-)N_-$$
$$- (r - \dot{U} - 2\sqrt{3}N_\times)\Sigma_\times \;, \tag{1g}$$

and

$$0 = (\boldsymbol{\partial}_1 - r + \dot{U})(1 + \Sigma_+) \tag{2a}$$

$$0 = 1 - (\Sigma_+^2 + \Sigma_2^2 + \Sigma_-^2 + N_\times^2 + \Sigma_\times^2 + N_-^2) \tag{2b}$$

$$0 = (1 + \Sigma_+)\dot{U} + 3(N_\times \Sigma_- - N_- \Sigma_\times) \tag{2c}$$

$$0 = (\boldsymbol{\partial}_1 - r + \sqrt{3}N_\times)\Sigma_2 \;, \tag{2d}$$

where $q := 2(\Sigma_+^2 + \Sigma_-^2 + \Sigma_\times^2 + \Sigma_2^2) - \frac{1}{3}(\boldsymbol{\partial}_1 - r + \dot{U})\dot{U}$. Since we are concerned with generic features, we restrict to the case $\Sigma_+ \neq -1$ ($\Sigma_+ = -1$ yields the Minkowski spacetime). We use the gauge constraint (2a) and the Codacci constraint (2c) to solve for r and $\dot{U}$ and so obtain the reduced state vector $\boldsymbol{X} = (E_1{}^1, \Sigma_+, \Sigma_2, \Sigma_-, N_\times, \Sigma_\times, N_-) = (E_1{}^1) \oplus \boldsymbol{S}$. Note that the Gauß constraint (2b) implies that the components of $\boldsymbol{S}$ are bounded.

Because of the symmetry restrictions, $E_1{}^1$ is the only spatial frame variable in our state space; in the present context asymptotic silence is thus associated with $E_1{}^1 \to 0$, while $E_1{}^1 = 0$ is referred to as the silent boundary. Our numerical experiments, which employ the `RNPL` [10] and `CLAWPACK` [11] packages with up to 2^{16} spatial grid points on the x-interval $(0, 2\pi)$, indicate that asymptotic silence holds in the present G_2 case for *all* timelines of a generic solution. Indeed, our numerical simulations indicate that $\max_x(E_1{}^1)$ decays exponentially. Moreover, they indicate that $\lim_{t \to +\infty}(\Delta := ||E_1{}^1 \partial_x \boldsymbol{S}||^2) = 0$ along generic timelines of a generic solution, i.e., generically the singularity is asymptotically silent and local, and hence in this case the asymptotic dynamics is governed by the equations on the silent boundary.

On the silent boundary $E_1{}^1 = 0$, the integrability condition and Eq. (1e) yield $r^2 = -3fN_\times^2$, while Eq. (2d) reduces to $0 = (r - \sqrt{3}N_\times)\Sigma_2$. In contrast to Ref. [4], we are here concerned with the general case $\Sigma_2 \neq 0$, for which $r = \sqrt{3}N_\times$ and hence $f = -1$; in this case the equations on the silent boundary are identical to the Hubble-normalized equations of the exceptional SSS Type$-_1\text{VI}_0$ models; see Wu [12, p. 635].

Our numerical experiments suggest that, in addition to $E_1{}^1 \to 0$, $\boldsymbol{C} := (\dot{U}, r, N_\times, N_-\Sigma_\times) \to 0$ holds for generic timelines of a generic solution when $t \to +\infty$. *On the silent boundary* $E_1{}^1 = 0$, $\boldsymbol{C} = 0$ yields the Kasner and SH Type–II subsets which are defined by $0 = E_1{}^1 = N_- = N_\times = \dot{U} = r$, $1 = \Sigma_+^2 + \Sigma_-^2 + \Sigma_\times^2 + \Sigma_2^2$, $q = 2$ and $0 = E_1{}^1 = \Sigma_\times = N_\times = \dot{U} = r$, $q = 2(1 - N_-^2)$, respectively.

With the present gauge choices, the Kasner subset contains a subset of equilibrium points: $0 = E_1{}^1 = \Sigma_2 = N_\times = \Sigma_\times = N_- = \dot{U} = r$, $1 = \Sigma_+^2 + \Sigma_-^2$, the Kasner circle, $\mathcal{K}$, which plays an essential role for the asymptotic dynamics. A linear stability analysis of $\mathcal{K}$ shows that all variables are stable when $t \to +\infty$, except for $(N_-, \Sigma_\times, \Sigma_2)$ which obey:

$$N_- = \hat{N}_- e^{-[1-k(x)]t} \tag{3a}$$

$$\Sigma_\times = \hat{\Sigma}_\times e^{-k(x)t} \tag{3b}$$

$$\Sigma_2 = \hat{\Sigma}_2 e^{[3-k(x)][1+k(x)]t/4} \;; \tag{3c}$$

$E_1{}^1$ and $\boldsymbol{C}$ decay exponentially and uniformly $[N_-\Sigma_\times \propto \exp(-t)]$. Here "hatted" variables are functions of x only. On $\mathcal{K}$, $\Sigma_+ = \hat{\Sigma}_+$, $\Sigma_- = \hat{\Sigma}_-$, and $k(x) := -\sqrt{3}\hat{\Sigma}_-(x)/[1 + \hat{\Sigma}_+(x)]$. Thus, N_-, $\Sigma_\times$, Σ_2 are unstable when $k(x) > 1$, $k(x) < 0$, $-1 < k(x) < 3$, respectively; see Fig. 1(a). The unstable mode N_- induces physical curvature transitions, associated with the SH Type–II subset on $E_1{}^1 = 0$, while $\Sigma_\times$ and Σ_2 induce frame transitions that lead to rotations of the spatial frame and multiple representations of the same solution, see Fig. 1(b); nevertheless, for the present frame choice it is these gauge transitions that make repeated curvature transitions possible, and hence they have indirect
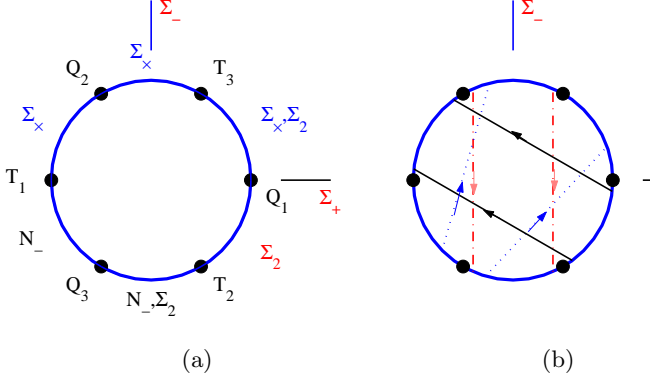
FIG. 1: (a) Unstable variables on $\mathcal{K}$, and (b) single transition sets associated with N_- (dotted), $\Sigma_\times$ (dash-dotted) and Σ_2 (solid).
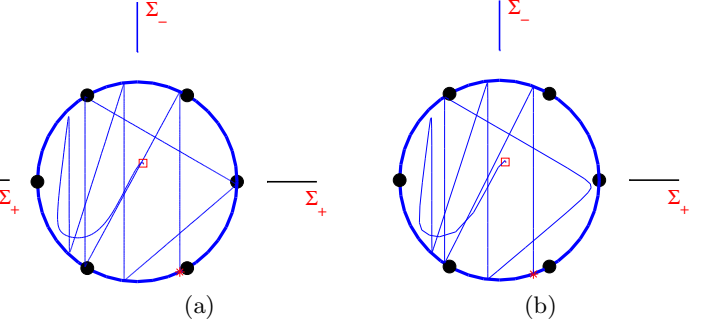

FIG. 2: Projections onto the $(\Sigma_+\Sigma_-)$-plane of a state space orbit along the typical timeline $x = 0.3$ for (a) the full G_2 system, and (b) its restriction to the silent boundary. In both cases the orbits approach $\mathcal{U}_{\text{vac}}^-$, i.e., $\Delta \to 0$ in (a), and $(E_1{}^1, \boldsymbol{C}) \to 0$ in both (a) and (b).

physical implications. The N_-, $\Sigma_\times$ and Σ_2 transitions imply that $k(x)$ changes according to the rules $k \to 2-k$, $k \to -k$ and $k \to (k+3)/(k-1)$, respectively.

The variables and equations that describe $\mathcal{U}_{\text{vac}}^-$ on $E_1{}^1 = 0$ and the exceptional SH Type–VI$^*_{-1/9}$ case, as given by Hewitt *et al.* [13], are identical. As shown in Sec. 5 of Ref. [13], there exist two integrals that describe the transition orbits. Although multiple transitions are possible, single transitions increasingly dominate. However, since frame transitions constitute gauge effects we will not pursue this further. What is important physically is that the variety $\mathcal{U}_{\text{vac}}^-$ induces an infinite sequence of Kasner states related by SH Type–II curvature transitions according to the frame invariant BKL map: $u \to u - 1$, if $u \geq 2$, and $1/(u - 1)$, if $1 < u < 2$, where u is defined frame invariantly by $\det \Sigma_{\alpha\beta} = \frac{1}{3}\Sigma_\alpha{}^\beta\Sigma_\beta{}^\gamma\Sigma_\gamma{}^\alpha = 2 - 27u^2(1+u)^2/(1+u+u^2)^3$.

Numerical investigations of vacuum SH Type–VI$^*_{-1/9}$ and SSS Type–$_{-1}$VI$_0$ models indicate that generic solutions asymptotically approach $\mathcal{U}_{\text{vac}}^-$. Our investigation suggests that this is also true for the evolution associated with generic timelines of the present inhomogeneous vacuum G_2 cosmologies, since our numerical results indicate that $(E_1{}^1, \boldsymbol{C}, \Delta) \to 0$ when $t \to +\infty$ for a generic timeline, see Fig. 2, and that thus the BKL map holds asymptotically for such a timeline.

Belinskiĭ [14] expressed concern that spatial structure, created by the effect of different timelines going through transitions at different times, could cause problems for the BKL scenario. Numerical experiments show that this is not the case for generic timelines. The reason is that spatial structure, created by the mechanism described above, develops on superhorizon scales; $\Delta \to 0$ within the shrinking particle horizons of a generic timeline when $t \to +\infty$.

Our investigations indicate that $E_1{}^1 \to 0$ as $t \to +\infty$ for all timelines of a generic solution, and that $(\boldsymbol{C}, \Delta) \to 0$ for generic timelines so that $\mathcal{U}_{\text{vac}}^-$ is a local past attrac-

tor. However, there are indications that spiky features, closely related to the spikes in Gowdy vacuum spacetimes, form along exceptional timelines; for such timelines neither $\boldsymbol{C}$ nor Δ has a limit.

Recall that for the present general G_2 case, the whole of $\mathcal{K}$ is unstable with respect to at least one of N_-, $\Sigma_\times$, Σ_2; see Eqs. (3). As in the Gowdy case, spike formation is caused by the occurrence of a zero for one of these variables at a point $(t, x(t))$ when the system is close to $\mathcal{K}$. It follows from Eqs. (1c) and (2d) that, for a generic smooth solution, Σ_2 cannot cross zero and thus produces no spikes. Spikes in $\Sigma_\times$ are "false" (gauge) spikes, while spikes in N_- are "true" (physical) effects which yield inherently inhomogeneous dynamical features in the Hubble-normalized Weyl curvature scalars. Linear analysis at $\mathcal{K}$ shows $E_1{}^1 \propto \hat{E}_1{}^1 e^{-t}$ and

$$E_1{}^1\partial_x N_- \propto \hat{E}_1{}^1\left[\partial_x\hat{N}_- + t\,\hat{N}_-\,\partial_x k(x)\right] e^{-[2-k(x)]t} \ .$$

The state space orbits of the spatial points outside the particle horizon of $(t, x(t))$, defined by $N_-(t, x(t)) = 0$, undergo curvature transitions with $N_- = \mathcal{O}(1)$ and opposite signs on either side of $x(t)$ when $k(x) > 1$; since $x(t)$ does not go through such a transition this leads to the formation of a spike. For $k(x) > 2$, $\boldsymbol{\partial}_1 N_- = E_1{}^1\partial_x N_-$ is unstable on $\mathcal{K}$, and hence grows in modulus at $(t, x(t))$, which leads to a growth in modulus of $\Sigma_\times$. Since the particle horizon size at $(t, x(t))$ is of order $E_1{}^1$, see UEWE, the above implies that $\partial_x N_-$ grows to order $1/E_1{}^1$ and that $\boldsymbol{\partial}_1 N_-$, and thus also Δ, is then $\mathcal{O}(1)$ at $(t, x(t))$. It therefore follows that the dynamics fails to be local. Moreover, one can similarly argue that $\boldsymbol{C}$ becomes $\mathcal{O}(1)$. Since the dynamics fails to be local at $(t, x(t))$, it is not governed by the silent boundary dynamical system. Nevertheless, our investigations indicate that the asymptotic dynamics is quite simple, and that it is related to that on the silent boundary. Numerical simulations show that the orbit described by $\boldsymbol{S}(t, x(t))$ during the formation and smoothing out of a spike is described by the map
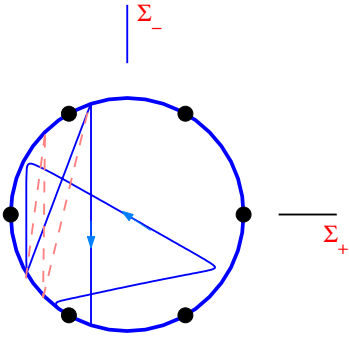
FIG. 3: Projection onto the $(\Sigma_+\Sigma_-)$-plane of a state space orbit undergoing a spike transition, followed by Σ_2 and $\Sigma_\times$ induced frame transitions and another spike transition (solid). The combination of N_- curvature and $\Sigma_\times$ frame transitions corresponding to the second spike transition is shown (dashed). See also Ref. [15, p. 152].

$k \to 4 - k$, equivalent to a sequence of local N_-–$\Sigma_\times$–N_- transitions; following Ref. [15], we refer to this behavior as a *spike transition*, see Fig. 3. The simplicity of this structure suggests that there may exist an effective dynamical system governing the spike transitions, playing a role analogous to that of the silent boundary system.

Numerical investigations suggest that an isolated zero in N_- may persist as $t \to +\infty$; if true this yields an infinite sequence of recurring spike transitions. Since the horizon scale decays exponentially, one expects $x(t)$ to converge exponentially to a point x_{spike}, and since the dynamics fails to be local during spike transitions, it follows that $\boldsymbol{C}$ and Δ fail to have limits along the timeline $x = x_{\mathrm{spike}}$; we refer to such a timeline as a *spike timeline*. Timelines along which $(\boldsymbol{C}, \Delta) \to 0$ as $t \to +\infty$ are called *non-spike timelines*. Since the opportunities for new spike formation occur at increasing time intervals due to the fact that $\mathcal{K}$ consists of equilibrium points for the system, and since the horizon size decreases exponentially, we conjecture that generic timelines are non-spike timelines. Our analysis supports the conjecture that as $t \to +\infty$ the Kretschmann scalar becomes unbounded, also along spike timelines.

For the Gowdy case ($\Sigma_2 = 0$), the sequence of spike transitions terminates when $k(x(t))$ reaches the interval $(0, 2)$ on the lower part of $\mathcal{K}$; see also Ref. [16]. If $k(x(t))$ reaches the interval $(0, 1)$, the spike disappears completely, while in the interval $(1, 2)$ a permanent spike is formed for which $\Delta \to 0$. Hence, for the Gowdy case, $\Delta \to 0$ uniformly, and thus the singularity is asymptotically silent and local for *all* timelines.

The exponentially shrinking particle horizons of the spike timelines cause severe numerical difficulties. Another obstacle are the subsets associated with the Taub points T_α [cf. Fig. 1(a)]; in the present case T_1 in particular. For SH Type–IX models, studied by Ringström [17],

the system spends a dominant portion of its time undergoing oscillations in the vicinity of the Taub points; this can be expected to hold also in the present G_2 case. In G_0 cosmologies, recently studied numerically by Garfinkle [18] in terms of the framework of UEWE, these issues should cause formidable problems; their resolution is likely to constitute a major step toward a rigorous analysis of generic spacetime singularities, and an understanding of the cosmic censorship problem.

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